Amplitude Distribution of Shot Noise

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A shot noise, $I(t)$, is a superposition of impulses occurring at random Poisson distributed times $\cdots, t_{-1}, t_0, t_1, t_2, \cdots$. In the simplest case, if the impulses all have the same shape $F(t)$, then $I(t) = \sum_i F(t - t_i)$. We study, in this and more general cases, the distribution function $Q(I) = Pr[I(t) \leqq I]$. One of our results is an integral equation for $Q(I)$. This yields explicit expressions for $Q(I)$ in a number of cases, including $F(t) = e^{-t}$; it also permits a computational technique which is applied to $F(t) = e^{-t} \sin \omega t$ for $\omega \gg 1$.

I. INTRODUCTION

A shot noise, $I(t)$, is a superposition of impulses occurring at random times $\cdots, t_{-1}, t_0, t_1, t_2, \cdots$. If the impulses all have the same shape, $F(t)$, then

$$I(t) = \sum_i F(t - t_i). \tag{1}$$

More generally, the impulse shapes may be randomly chosen from a family of shapes, $F(a, t)$, depending on a parameter a. Then

$$I(t) = \sum_i F(a_i, t - t_i). \tag{2}$$

We assume that the times t_i form a Poisson sequence with rate n impulses per second. In the case of (2), we assume that the parameters a_i are chosen independently from a common distribution.

We study the amplitude distribution function

$$Q(I) = \Pr[I(t) \leqq I].$$

Rice[1] (Section 1.4) considered the noise (1) and noises (2) with $F(a, t) = aF(t)$. He expressed the density function $P(I) = Q'(I)$ as a Fourier

integral. The Fourier integral is difficult to evaluate except by means of a series that Rice derived for the case of noises that are nearly gaussian (large impulse rate n). In our treatment we derive for (1) an integral equation

$$\int_{-\infty}^{I} x\, dQ(x) = n \int_{-\infty}^{\infty} Q[I - F(t)]F(t)\, dt \tag{3}$$

or, equivalently,

$$IQ(I) = \int_{-\infty}^{I} Q(x)\, dx + n \int_{-\infty}^{\infty} Q[I - F(t)]F(t)\, dt. \tag{4}$$

We solve (3) for some special choices of $F(t)$ illustrated in Fig. 1. The section numbers on Fig. 1 give the part of the text in which each $F(t)$

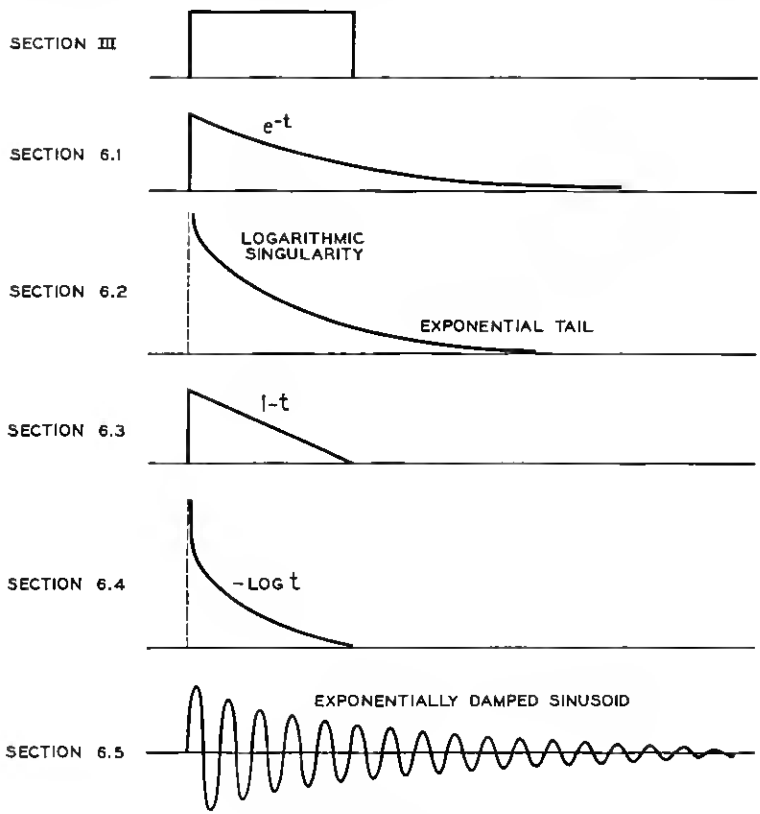

Fig. 1 — Impulse functions $F(t)$.

is considered. Analytic solutions are obtained for all cases except the last one of these; this case is an important one in practice, and so was chosen to illustrate a numerical solution of (3).

For purposes of finding $Q(I)$ any given noise of form (2) can be replaced by an equivalent one of form (1). This equivalence is discussed in Section III.

Several different ways of deriving (3) are possible. We give an analytic proof (Section IV) and a probabilistic proof (Section V).

II. CHARACTERISTIC FUNCTIONS

All impulse functions $F(t)$, $F(a, t)$ will be assumed integrable over $-\infty < t < \infty$. This assumption is no practical restriction and is made to ensure that the series (1) and (2) converge.

We begin by deriving the characteristic function $C(s)$ of $I(t)$ for the noise (2); i.e.,

$$C(s) = E[e^{-sI(t)}]. \tag{5}$$

Here E denotes the expected value.

The characteristic function $C_0(s)$ in the case of the noise (1) is

$$C_0(s) = \exp\left\{-n \int_{-\infty}^{\infty} (1 - \exp[-sF(t)])\, dt\right\} \tag{6}$$

(see Ref. 1, Section 1.4–7). We may obtain $C(s)$ directly from (6) by regarding the noise (2) as a superposition of noises of the form (1) with different choices of $F(t)$. Suppose, for example, that the parameter a has only a discrete range of possible values ($a = A_1, A_2, A_3, \cdots$), and let p_k be the probability of picking a to be A_k. Then, collecting together the terms of (2) for which a_i has the same value, one expresses $I(t)$ as a sum of new independent random variables

$$I(t) = I_1(t) + I_2(t) + \cdots,$$

where $I_k(t)$ is a noise of the form (1) in which the impulse shape is $F(t) = F(A_k, t)$ and the impulses arrive at an average rate, np_k per second. If $C_k(s)$ is the characteristic function of $I_k(t)$,

$$C(s) = E[e^{-s(I_1(t)+I_2(t)+\cdots)}],$$

$$C(s) = C_1(s)C_2(s)\cdots. \tag{7}$$

In (7) each $C_k(s)$ may be evaluated by an expression of the form (6), and the final result is

$$C(s) = \exp\left[-n\left(\int_{-\infty}^{\infty} \{1 - E[e^{-sF(a,t)}]\}\, dt\right)\right]. \tag{8}$$

In (8) the expectation E is taken with respect to the random parameter a. Altbough our derivation of (8) used the assumption that a had a discrete range of values, a convincing limiting argument can be given for the truth of (8) in general. Alternatively one can rederive (8) in general by a slight modification of Rice's derivation of (6).

In a similar way, we find

$$E\,\{\exp\,[-\,s_1 I(\tau_1) - \cdots - s_N I(\tau_N)]\}$$
$$= \exp\left\{-n \int_{-\infty}^{\infty} 1 - E\left[\exp - \sum_{k=1}^{N} s_k F(a, t - \tau_k)\, dt\right]\right\},$$

which might be used to study the joint distribution of $I(\tau_1), \cdots, I(\tau_N)$.

III. EQUIVALENCE

In (6) it is evident that there are many different ways of choosing an $F(t)$ to obtain the same distribution of $I(t)$. The integral in (6) remains unchanged if $F(t)$ is replaced by any other function $F_0(t)$ such that, for every choice of u_1 and u_2, the two sets S and S_0 of times t that are defined by:

$$\begin{aligned} S: &\quad u_1 < F(t) \leqq u_2\,, \\ S_0: &\quad u_1 < F_0(t) \leqq u_2 \end{aligned} \tag{9}$$

have the same measure: For example, the second function in Fig. 1 may be replaced by $e^{-2|t|}$. Some idea of the freedom with which one can construct such a new $F_0(t)$ from a given $F(t)$ may be had from Fig. 2. Given $F(t)$ and n, one can construct a measure, $dg(u)$, on the real u line by defining the measure of the interval $u_1 < u \leqq u_2$ to be n times the Lebesgue measure of the set of times t for which $u_1 < F(t) \leqq u_.$. Then, changing the variable of integration in (6) from t to u, one obtains

$$C_0(s) = \exp\left[-\int (1 - e^{-su})\, dg(u)\right]. \tag{10}$$

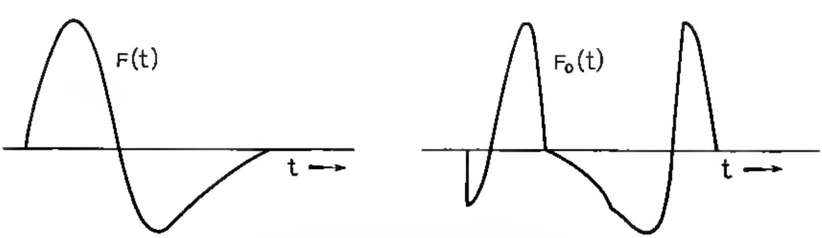

Fig. 2 — A pair of equivalent impulse functions.

Similarly, in the case of noise (2), let $dg(u)$ be defined in such a way that the interval $u_1 < u \leqq u_2$ has measure equal to n times the expected value of the Lebesgue measure of the random set of times t satisfying

$$u_1 < F(a, t) \leqq u_2 . \tag{11}$$

Changing the variable of integration in (8) from t to u, one obtains for $C(s)$ an expression that is just the right-hand side of (10). Thus, we call two noises which have the same $dg(u)$ *equivalent* and have shown

Theorem: The distributions of the amplitudes $I(t)$ of equivalent noises are the same.

Given a noise (2) with measure $dg(u)$, one can find a function $F(t)$ such that

$$F\left(\int_u^\infty \frac{dg(w)}{n}\right) = u \qquad \text{for } u > 0 \tag{12}$$

and

$$F\left(-\int_{-\infty}^u \frac{dg(w)}{n}\right) = u \qquad \text{for } u < 0. \tag{13}$$

The noise (1) with this choice of $F(t)$ is equivalent to the given noise (2). Then every noise (2) is equivalent to a noise of form (1). For the problem of finding the amplitude distribution we now need consider only noises of the form (1).

As a very simple application of the theorem, consider (2) with the family of impulse functions

$$F(a, t) = \begin{cases} b & \text{if } 0 \leqq t \leqq a \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where the parameter a is distributed over positive values only. To find the measure $dg(u)$, note that the set (11) has Lebesgue measure

$$\int_{u_1}^{u_2} dg(u) = \begin{cases} \infty & \text{if } u_1 < 0 \leqq u_2 \\ a & \text{if } 0 \leqq u_1 < b \leqq u_2 \\ 0 & \text{otherwise.} \end{cases}$$

Then, $dg(u)$ must lump all its measure onto two points $u = 0$ and $u = b$. The measure of 0 is ∞ and the measure of b is nA, where $A = E(a)$, the expected length of the pulse (14). A noise of the form (1) that also has the measure $dg(u)$ is the one with

$$F(t) = \begin{cases} b & \text{if } 0 \leqq t \leqq A \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

This happens to be a noise for which the amplitude distribution is easily obtained. Since $I(t)$ is just the number of impulses which arrive in the time interval from $t - A$ to t, $I(t)$ has the Poisson distribution

$$\Pr\,[I(t) = kb] = \frac{(nA)^k}{k!}\,e^{-nA}.$$

By the theorem, this result also solves the distribution problem for the original noise (14). In the same way, the example may be generalized as follows: If $F(a, t)$ has the form $S(t/a)$ for some given impulse function $S(t)$, then the amplitude $I(t)$ in (2) has the same distribution as the amplitude in (1) where $F(t)$ is taken to be $S[t/(E \mid a \mid)]$.

Although the measure $dg(u)$ determines the amplitude distribution, it does not determine all the statistical properties of the noise. One can easily find examples of functions $F(t)$ and $F_0(t)$ for which the corresponding noises (1), although equivalent, have different joint distributions for the pair of random variables $I(\tau_1)$, $I(\tau_2)$. The spectrum of $I(t)$ is proportional to the squared magnitude of the Fourier transform of $F(t)$ (see Campbell's theorem in Ref. 1, Sections 1.2, 1.3), and so can be changed without changing $dg(u)$.

IV. DERIVATION

A proof of (3) can be given from the formula (10) for the characteristic function $C(s)$. We will assume here that the impulses $F(t)$ or $F(a, t)$ in question are nonnegative functions. This restriction will be removed in the next section and is made now in order to allow Laplace transform methods to be used.

We have

$$\begin{aligned} C(s) &= E[e^{-sI(t)}] \\ &= \int_{-0}^{\infty} e^{-sI}\,dQ(I) \\ &= s\int_0^{\infty} e^{-sI}Q(I)\,dI, \end{aligned}$$

where the last formula is obtained by integrating by parts and noting $Q(0-) = 0$. Thus, $C(s)/s$ is the Laplace transform of $Q(I)$.

Using (10),

$$\begin{aligned} \frac{C'(s)}{C(s)} &= \frac{d}{ds}\log C(s) \\ &= -\int e^{-su}u\,dg(u). \end{aligned}$$

Then

$$-\frac{C'(s)}{s} = \frac{C(s)}{s} \int e^{-su} u \, dg(u). \tag{16}$$

Take the inverse Laplace transform of (16). The product on the right of (16) transforms into a convolution. One of the terms, $C(s)/s$, is the Laplace transform of $Q(I)$. The other term is already in the form of a Laplace transform. Then the product transforms into

$$\int_0^I Q(I - u) u \, dg(u).$$

This integral is another way of writing

$$\int Q[I - F(t)] F(t) \, dt.$$

To prove (3), it now remains to show that the term $-C'(s)/s$ in (16) is the Laplace transform of

$$\int_0^I x \, dQ(x).$$

This follows because

$$C'(s) = -\int_0^\infty e^{-sI} I \, dQ(I)$$

and because $1/s$ is the Laplace transform of the unit step function.

The integral equation (3) can be proved in several other ways; in particular, a more probabilistic proof will be obtained as a hyproduct in the next section.

V. SUMS OF NOISES

Let $I_1(t)$ and $I_2(t)$ be two independent shot noises with impulse responses $F_1(t)$, $F_2(t)$, impulse rates n_1, n_2 and measures $dg_1(u)$, $dg_2(u)$. Their sum $I(t) = I_1(t) + I_2(t)$ is a shot noise of the form (2). The impulse rate for the sum $I(t)$ is $n = n_1 + n_2$; the function $F(a, t)$ is chosen to be $F_1(t)$ or $F_2(t)$ with probabilities n_1/n and n_2/n. The measure $dg(u)$ for $I(t)$ is

$$dg(u) = dg_1(u) + dg_2(u).$$

Since $I(t)$ is the sum of the independent random variables $I_1(t)$ and

$I_2(t)$, the distribution of $I(t)$ can be obtained from those of $I_1(t)$ and $I_2(t)$ by a convolution.

This observation may be used occasionally to compute the distribution function for a given noise $I(t)$. Suppose the measure $dg(u)$ of $I(t)$ can be written as a sum $\Sigma\, dg_i(u)$ of measures of noises $I_i(t)$ for which the distribution functions are known. Then $I(t)$ is equivalent to a sum of independent noises $\Sigma\, I_i(t)$ and its distribution function may be obtained by convoluting the distributions of $I_i(t)$ together. Even when an exact decomposition $dg(u) = \Sigma\, dg_i(u)$ is not known one might approximate $dg(u)$ by a sum to get an approximate $Q(I)$.

A particular instance of a decomposition is the following. Let measures $dg^+(u)$ and $dg^-(u)$ be defined by

$$dg^+(u) = \begin{cases} dg(u) & \text{if } u > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$dg^-(u) = \begin{cases} dg(u) & \text{if } u \leqq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $dg(u) = dg^+(u) + dg^-(u)$. At present, our proof of (3) holds only for positive noises. A similar derivation of (3) for negative noises also holds. In the general case we might consider our noise $I(t)$ to be equivalent to a sum $I^+(t) + I^-(t)$ of independent positive and negative noises, and compute $Q(I)$ as a convolution

$$Q(I) = \int Q^+(I - x)\, dQ^-(x).$$

That (3) holds in general now follows from the next lemma.

Lemma: Let $dg_1(u)$, $dg_2(u)$, $dg(u)$ be measures such that

$$dg(u) = dg_1(u) + dg_2(u).$$

Let $Q_1(I)$, $Q_2(I)$ be solutions of the integral equation corresponding to the measures $dg_1(u)$, $dg_2(u)$. Then for measure $dg(u)$ the convolution

$$Q(I) = \int Q_1(I - w)\, dQ_2(w)$$

$$= \int Q_2(I - x)\, dQ_1(x)$$

is a solution of the integral equation (3).

Proof:

$$\begin{aligned}
\int_{-\infty}^{I} y\,dQ(y) &= \int_{-\infty}^{I}\int_{w} y\,dQ_1(y-w)\,dQ_2(w) \\
&= \int_{-\infty}^{I}\int_{w} [(y-w)+w]\,dQ_1(y-w)\,dQ_2(w) \\
&= \int_{-\infty}^{\infty}\int_{w} Q_1(I-w-u)u\,dg_1(u)\,dQ_2(w) \\
&\quad + \int_{-\infty}^{\infty}\int_{w} dQ_1(y-w)Q_2(I-w)u\,dg_2(u) \\
&= \int_{-\infty}^{\infty} Q(I-u)u\,dg(u).
\end{aligned}$$

Taking dg_1 and dg_2 as dg^+ and dg^-, the lemma, together with the result of Section IV, completes the derivation of (3).

A different proof of (3) may now be outlined as follows. Consider first the noise with response function (15). The integral equation (3) holds in this special case. For $dg(u)$ gives measure nA to the point $u = b$ and

$$Q(I) = \sum_{kb \leqq I} \frac{(nA)^k}{k!} e^{-nA}.$$

Then

$$\begin{aligned}
\int_{-\infty}^{I} x\,dQ(x) &= \sum_{kb \leqq I} kb\,\frac{(nA)^k}{k!} e^{-nA} \\
&= nAbQ(I-b) \\
&= \int_{-\infty}^{\infty} Q(I-u)u\,dg(u).
\end{aligned}$$

Next, any step function $F(t)$ with a finite number of steps has a measure $dg(u) = \Sigma\, dg_i(u)$, where each $dg_i(u)$ concentrates its measure on a single value of u [a level of $F(t)$]. We have just proved that the integral equation holds for each of the $dg_i(u)$ noises. By the lemma we can conclude that (3) holds for all step function noises. By limiting arguments, one might establish (3) more generally.

VI. EXAMPLES

For certain choices of $F(t)$ the integral equation can be solved easily. Some special cases of this kind will be examined in this section.

6.1 *Example 1*

First consider a noise (1) with

$$F(t) = \begin{cases} e^{-t} & \text{if } t \geqq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

We expect a distribution $Q(I)$ that has a density $P(I) = Q'(I)$ and so write the integral equation in the form

$$IP(I) = n \int_0^1 P\,(I - u)\, du = n \int_{I-1}^{I} P(x)\, dx.$$

Differentiating, we obtain

$$IP'(I) - (n - 1)P(I) = -nP(I - 1).$$

To solve this differential difference equation, note first that, when $0 \leqq I < 1$, $P(I - 1) = 0$. Hence, for $0 \leqq I < 1$,

$$P(I) = cI^{n-1},$$

where c is a constant of integration to be determined. For larger values of I, the differential difference equation may be converted to an integral form

$$P(I) = I^{n-1}\left[c - n \int_1^I P(x - 1)x^{-n}\, dx\right].$$

Since the integrand is known for $x < 2$, we can determine $P(I)$ for $I < 2$. Next, this result enables us to integrate further to get $P(I)$ for $I < 3$, etc. Clearly, the analytic form of $P(I)$ changes at $I = 1, 2, 3, \cdots$. For example, when $n = 1$, we have

$$\begin{aligned} P(I) &= c && \text{if } 0 \leqq I \leqq 1, \\ P(I) &= c(1 - \log I) && \text{if } 1 \leqq I \leqq 2, \\ P(I) &= c\left[1 - \log I + \int_2^I \frac{\log\,(x-1)}{x}\, dx\right] && \text{if } 2 \leqq I \leqq 3, \end{aligned}$$

etc.

Finally, the constant c must be determined by the condition

$$\int_0^\infty P(I)\, dI = 1.$$

The constant can be determined as follows: The Laplace transform

$\hat{p}(s)$ of $P(I)$ is the characteristic function. By (6),

$$\hat{p}(s) = \exp\left(-n \int_0^s \frac{1 - e^{-y}}{y}\, dy\right).$$

This may be rewritten, with the aid of partial integration, as

$$\hat{p}(s) = \exp\left[-n(1 - e^{-s}) \log s + n\int_0^\infty e^{-y} \log y\, dy - n \int_s^\infty e^{-y} \log y\, dy\right]$$

$$= s^{-n} e^{-n\gamma}\{1 + 0[e^{-s(1-\epsilon)}]\} \qquad \text{for any } \epsilon > 0.$$

Thus, for $0 < I < 1$,

$$P(I) = \frac{e^{-n\gamma}}{\Gamma(n)} I^{n-1},$$

where $\gamma = 0.577215665\cdots$ is Euler's constant. Hence

$$c = \frac{e^{-n\gamma}}{\Gamma(n)}.$$

6.2 *Example 2*

Our next example concerns a noise (1) with $F(t)$ defined as follows:

$$F(t) = 0 \qquad \text{for } t \leqq 0,$$

and

$$t = \int_{F(t)}^\infty \frac{e^{-y}\, dy}{y} \qquad \text{for } t > 0.$$

This somewhat artificial noise interests us because it has a very simple $P(I)$. A sketch of $F(t)$ is shown in Fig. 1, marked "Section 6.2". For small t, $F(t)$ grows large but only logarithmically:

$$F(t) \sim -\log t.$$

For large t, $F(t)$ has an exponential tail,

$$F(t) \sim \gamma^t,$$

where γ is Euler's constant.

For this noise, $dg(u) = ne^{-u}\, du$ and the integral equation can be put in the form

$$IP(I)e^I = n \int_0^I P(x)e^x\, dx.$$

Differentiating, we obtain a very simple differential equation and the solution is

$$P(I) = \frac{e^{-I} I^{n-1}}{\Gamma(n)} .$$

This solution exhibits a rapid approach of $P(I)$ to gaussian form (with mean n and variance n) as $n \to \infty$.

6.3 *Example 3*

If we consider the case

$$F(t) = \begin{cases} 1 - t & 0 \leqq t \leqq 1 \\ 0 & \text{otherwise,} \end{cases}$$

then the distribution $Q(I)$ has a jump of e^{-n} at $I = 0$, since e^{-n} is the probability that no point of the Poisson process falls into an interval of unit length. We therefore seek a density function $P(I)$ such that

$$Q(I) = e^{-n} + \int_0^I P(x)\,dx \qquad \text{for } I \geqq 0.$$

The integral equation then becomes

$$IP(I) = \begin{cases} n \int_0^I P(I - y)y\,dy + nIe^{-n} & \text{if } 0 < I \leqq 1 \\ n \int_0^1 P(I - y)y\,dy & \text{if } 1 < I. \end{cases}$$

If $P(I) = R''(I)$, this becomes

$$IR''(I) - nR(I) = \begin{cases} 0 & \text{if } I < 1 \\ -n[R'(I - 1) + R(I - 1)] & \text{if } I \geqq 1. \end{cases}$$

This can be solved recursively. In the first interval $(I < I)$

$$P(I) = \frac{nR(I)}{I} = \sqrt{n}\, e^{-n} \frac{I_1(2\sqrt{nI})}{\sqrt{I}} ,$$

where I_1 is the Bessel function, and where the coefficient of I_1 has been determined by substitution in the integral equation.

In the general case, if $I > 1$,

$$P(I) = e^{-n} \sum_{k=0}^{[I]} \frac{(-1)^k}{k!} n^{(k+1)/2}(I - k)^{(k+1)/2} I_{k-1}[2\sqrt{n(I - k)}],$$

where $[I]$ is the largest integer $\leqq I$. For example, if $1 < I \leqq 2$,

$$P(I) = e^{-n}\left[\sqrt{\frac{n}{I}}\, I_1(2\sqrt{nI}) - nI_0(2\sqrt{n(I-1)})\right].$$

(Note that $I_{-1} = I_1$.) These formulas can be derived either from the integral equation, or, with rather more courage but actually less work, from (6).

6.4 *Example 4*

Choosing

$$F(t) = \begin{cases} -\log t, & 0 < t \leqq 1 \\ 0 & \text{otherwise} \end{cases}$$

provides another simple case. Again, $\Pr(I = 0) = e^{-n}$, and we seek a density $P(I)$, as in Example 3. For values $I > 0$, $P(I)$ satisfies

$$IP(I) = n\int_0^I P(I-u)ue^{-u}\,du + ne^{-n}Ie^{-I}.$$

Now, letting $R(I) = Ie^{I}P(I)$, we obtain again the differential equation $IR''(I) + nR(I) = 0$, this time for all $I > 0$. The solution is

$$P(I) = e^{-(I+n)}\sqrt{\frac{n}{I}}\, I_1(2\sqrt{nI}) \qquad \text{for } I > 0.$$

6.5 *Example 5*

In this example we let $F(t)$ be the response of a simple tuned circuit to an impulse; i.e.

$$F(t) = e^{-t}\sin\left(\frac{2\pi t}{h}\right). \tag{17}$$

Although the period h appears in $F(t)$ as a parameter, the corresponding measure $dg_h(u)$ tends to a limiting measure $dg(u)$ as $h \to 0$. We will solve the integral equation numerically in this limiting case only. We then expect this result to be applicable as a good approximation whenever the tuned circuit has high Q; i.e., $h \ll 1$.

To get the limiting measure $dg(u)$ let us examine $F(t)$ in a small neighborhood of a time $t = T$ at which the sinusoid is at a maximum. During the period from T to $T + h$ the exponential e^{-t} changes only

by a factor $e^{-h} = 1 + O(h)$. Thus, aside from terms of order $O(h)$, $F(t)$ is

$$F(t) = e^{-T} \cos\left[\frac{2\pi(t - T)}{h}\right]$$

in this period. Given a level $u > 0$, $F(t) \geqq u$ for a time

$$\frac{h}{\pi} \operatorname{arc} \cos (ue^{T})[1 + O(h)]$$

during the period $T \leqq t \leqq T + h$. For small h, we conclude that $F(t)$ lies above level u for a total amount of time approaching

$$\int_0^\infty \frac{dg(w)}{n} = \frac{1}{\pi} \int_0^{-\log u} \operatorname{arc} \cos (ue^{T})\, dT$$
$$= \frac{1}{\pi} \int_u^1 \frac{\operatorname{arc} \cos Z}{Z}\, dZ$$

as $h \to 0$. Similarly, setting the amplitude level u at a negative value, we obtain

$$\int_{-\infty}^{u} \frac{dg(w)}{n} = \frac{1}{\pi} \int_{|u|}^{1} \frac{\operatorname{arc} \cos Z}{Z}\, dZ.$$

The measure $dg(u)$ is now known. We wish to solve the integral equation in which

$$dg(u) = \begin{cases} \dfrac{n \operatorname{arc} \cos |u|\, du}{\pi |u|} & \text{if } |u| < 1 \\ 0 & \text{if } |u| \geqq 1. \end{cases} \tag{18}$$

In this case, we had to resort to numerical methods. The integral equation might be approximated directly by a system of linear algebraic equations. However, such an approximation would be troublesome in our case because the integral equation is homogeneous. Unless we could guarantee that the approximating system would have determinant zero, there would be no nontrivial solution at all. The procedure that follows avoids this difficulty.

Let measures $dg^+(u)$, $dg^-(u)$ be defined as in Section V. We will solve the two integral equations with measures $dg^+(u)$ and $dg^-(u)$ and convolute the two solutions $P^+(I)$, $P^-(I)$ together to get the desired density $P(I)$. This approach has the advantage that the integral equation expresses $P^+(I)$ in terms of only values of P^+ for arguments $<I$. Thus, one can compute $P^+(I)$ approximately by a simple recurrence.

One might start the recurrence computation by assuming a value of $P^+(I)$ for a small I; afterward the solution could be normalized to make

$$\int_0^\infty P^+(I)\, dI = 1.$$

However, we can obtain in the Appendix an asymptotic formula for $P^+(I)$ for I near 0, and thereby start tbe recurrence with nearly correct values of $P^+(I)$.

Figs. 3 and 4 give the results of the computation for this limiting case of an infinitely rapidly oscillating tuned-circuit response. Computations were made for rates of 2.5, 5 and 10 impulses per unit time, where the time scale is determined by the exponential in (17). Fig. 3 shows $P(I)$ for these three cases; Fig. 4 plots $Q(I)$ on log normal paper and compares the result with the gaussian of the same mean and variance. As expected from Rice's theory, the noticeable differences are in the tails of the distributions; in the shot noise, very large values of I are *more* likely than in the corresponding limiting gaussian.

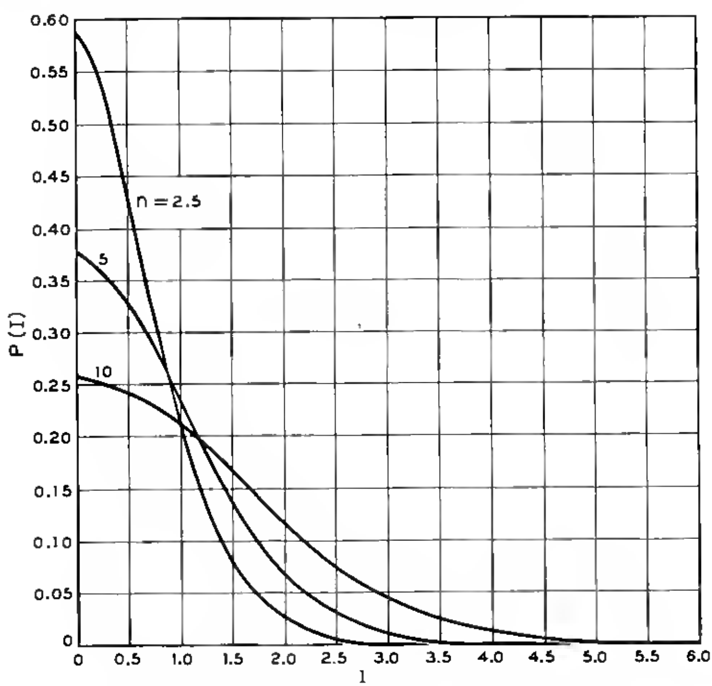

Fig. 3 — Amplitude density $P(I)$ for high-frequency damped sinusoid noise; $F(t) = e^{-t} \sin \omega t$, $\omega \gg 1$; $P(-I) = P(I)$.

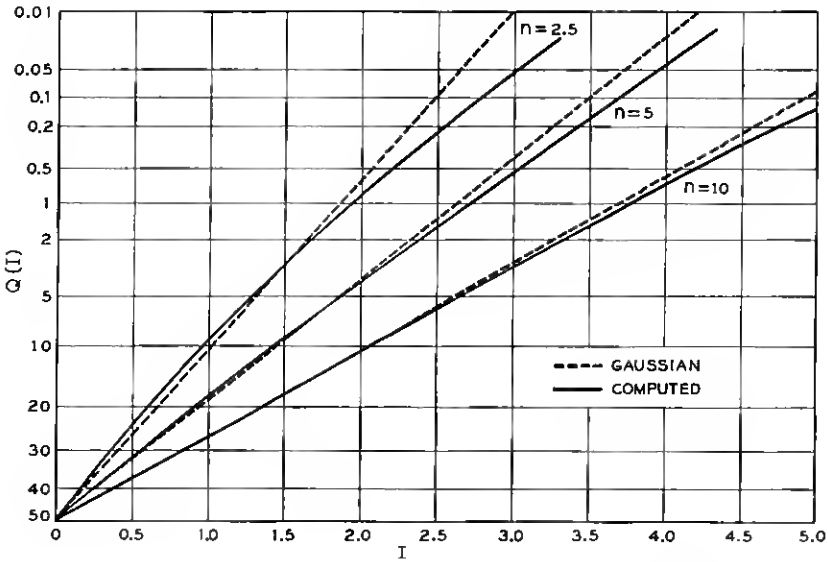

Fig. 4 — Comparison of amplitude distribution $Q(I)$ with gaussian approximation; $Q(-I) = 1 - Q(I)$.

As indicated above, the computation was performed in two stages. First, the integral equation for $P^+(I)$,

$$IP^+(I) = \int_0^I P^+(I - u)\, dg(u),$$

was solved by approximating the integral through Simpson's formula and solving the resulting triangular system of equations; the series for $P^+(I)$ near $I = 0$ was used to start the computation. Then, with the aid of the theory in Section V, $P(I)$ was computed as

$$P(I) = \int_0^\infty P^+(y)P^+(I + y)\, dy,$$

and $Q(I)$ as the integral of P.

VII. ACKNOWLEDGMENT

We wish, at this point, to thank Miss M. A. Lounsberry, for writing a general program for the IBM 704 for solving integral equations of this kind. Without her ingenuity and perseverance the present numerical results would not have been obtained.

APPENDIX

Derivation of the Form for $P^+(I)$ for Small I for a Tuned-Circuit Response

We begin with the integral equation

$$IP^+(I) = \frac{n}{\pi} \int_0^{\min(1,I)} P^+(I - u) \arccos u\, du.$$

Taking Laplace transforms, we obtain

$$\hat{p}'(s) + \frac{n}{2} \frac{\hat{p}(s)}{s} [1 - I_0(s) + L_0(s)] = 0,$$

where I_0 is the modified Bessel function, and L_0 the Struve function. Remembering the condition $\hat{p}(0) = 1$, and that $I_0(0) = 1$, $L_0(0) = 0$, we find

$$\hat{p}(s) = \exp\left[-\frac{n}{2} \int_0^s \frac{1 - I_0(u) + L_0(u)}{u}\, du \right].$$

Now $I_0 - L_0$ approaches zero at infinity rapidly enough for that portion of the integral to converge as $s \to \infty$, but the remaining term diverges. We thus write

$$\begin{aligned}\hat{p}(s) &= \exp\left[-\frac{n}{2} \int_0^s \frac{L_0(u) - I_0(u) + (u+1)^{-1}}{u} \right] du \\ &\quad + \frac{n}{2} \int_0^s \left[\frac{(u+1)^{-1}}{u} - \frac{1}{u} \right] du \\ &= (s+1)^{-n/2} \exp\left[-\frac{n}{2} \int_0^\infty \frac{L_0(u) - I_0(u) + (u+1)^{-1}}{u} \right] \\ &\quad \cdot \exp\left[\frac{n}{2} \int_s^\infty \frac{L_0(u) - I_0(u) + (u+1)^{-1}}{u}\, du \right].\end{aligned}$$

By Ref. 2, p. 426,

$$\frac{I_0(u) - L_0(u)}{u} = \frac{2}{\pi} \int_0^\infty \frac{J_0(x)}{x^2 + u^2}\, dx.$$

Hence,

$$\frac{I_0(u) - L_0(u) - (u+1)^{-1}}{u} = \frac{2}{\pi} \int_0^\infty \frac{J_0(x) - (1 + x^2)^{-1}}{x^2 + u^2}\, dx.$$

We may now integrate with respect to u and interchange integrations.

There results

$$\int_0^\infty \frac{I_0(u) - L_0(u) - (u+1)^{-1}}{u}\,du = \int_0^\infty \frac{J_0(x) - (1+x^2)^{-1}}{x}\,dx$$

$$= \int_0^\infty \left[J_1(x) - \frac{2x}{(1+x^2)^2}\right] \log x\,dx$$

$$= \log 2 - \gamma$$

by Gröbner and Hofreiter[3] and Bierens de Haan.[4] Thus,

$$p(s) = e^{-(n/2)\gamma}(2)^{n/2}(s+1)^{-n/2}$$

$$\cdot \exp\left[\frac{n}{2}\int_s^\infty \frac{L_0(u) - I_0(u) + (u+1)^{-1}}{u}\,du\right].$$

It remains only to discuss the behavior of the last integral for large s. By Ref. 2, p. 332,

$$L_0(u) - I_0(u) = -\frac{2}{\pi}\int_0^\infty \frac{\sin ux}{\sqrt{1+x^2}}\,dx.$$

Hence,

$$\int_s^\infty \frac{L_0(u) - I_0(u)}{u}\,du = -\frac{2}{\pi}\int_0^\infty \frac{dx}{\sqrt{1+x^2}}\int_{sx}^\infty \frac{\sin y}{y}\,dy$$

$$= \frac{2}{\pi}\int_0^\infty \log(x + \sqrt{1+x^2})\,\frac{\sin sx}{x}\,dx,$$

while

$$\int_s^\infty \frac{du}{u(u+1)} = \ln\left(1 + \frac{1}{s}\right).$$

Both of these are $O(1/s)$ for large s, and hence, for large s,

$$\hat{p}(s) = e^{-n/2\gamma}2^{n/2}s^{-n/2} + O[s^{-(n/2)-1}].$$

Hence

$$P^+(I) = \frac{2e^{-(n/2)\gamma}(2I)^{(n/2)-1}}{\Gamma\left(\frac{n}{2}\right)} + O(I^{n/2})$$

for small values of I.

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